HOMOGENIZATION OF TWO-PHASE FLOW: HIGH CONTRAST OF ONE PHASE PERMEABILITY

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The steady-state two-phase flow is simulated by a non-linear elliptic system of equations [3] that is a vector analogue of the non-linear thermal conductivity equation when the conductivity coefficient depends on "temperature". The classical homogenization procedure for non-linear equations ([2], [4]) gives the homogenized equation in the case when the only small parameter of the problem is equal to the ratio ϵ of the period of the microstructure to the characteristic size of the problem. This homogenized equation is of the same type as the initial one, i.e. if the steady-state flow equation for phase pressures $p_{i\epsilon}$ takes the form

$$div(\lambda_i(\frac{x}{\epsilon}, p_{1\epsilon} - p_{2\epsilon})\nabla p_{i\epsilon}) = f_i(x),\ i = 1, 2,\ x \in \boldsymbol{r}^s, s = 2, 3, \tag{1}$$

with 1-periodic in ξ coefficient $\lambda_i(\xi, P_c)$ and with f_i smooth enough, then the homogenized equation is $div(\widehat{\lambda}_{0i}(p_{10} - p_{20})\nabla p_{i0}) = f_i(x),\ i = 1, 2$, where $p_{i0}(i = 1, 2)$ are the macroscopic pressures and $\widehat{\lambda}_i$ are the macroscopic effective phase permeabilities calculated according to the standard [2] homogenization procedure.

Thus the macroscopic effective phase permeabilities depend on the difference of phase pressures but not on the gradients of these pressures. On the other hand some numerical experiments [5] show that $\widehat{\lambda}_{i0}$ **depend on these gradients, and the contribution of this dependency is of order of 1.**

The present paper explains this effect in the case when the model contains a second small parameter : the ratio of microscopic effective permeabilities of some low-permeable *for one of phases* zone occupying the domain G_2 and of the high-permeable *for the same phase* zone occupying G_1. It means that the material occupying G_1 is much more permeable than G_2 for one of phases while for the second phase their permeabilities are comparable.

This situation is realistic. For example, in some capillary pressure intervals the non-wetting phase can have permeability constrast of several orders of magnitude whereas for the wetting phase the permeability is of the same order in both high-permeable and the low-permeable zones. The discovered new effect was described in [6]; it may be important for the oil recovering engineering.

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EVOLUTION OF VORTEX KNOTS AND WINNING NUMBER EFFECT

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In this paper we address the study of the evolution of vortex torus knots and the influence of the winding number on the self-induced velocity in ideal fluid. This problem dates back to the original work of Lord Kelvin [1] on vortex atoms and it still presents challenging difficulties. A former study [2], based on the analytical solutions of steady torus knots under the localized induction approximation and the numer- ical evolution of these knots under the Biot-Savart law, has revealed unexpected stability features and strong coherency of these knot configurations, motivating further work. Here we consider the evolution of thin core vortex filaments in the shape of torus knots in the context of the Euler equations. Torus knots are twist knots embedded on a torus and are classified by their topological winding number, given by the ratio of the number of meridian to longitudinal turns. Since they may be specified by a relatively simple geometry, some analytical progress is possible. In the simple case of uniform core vorticity, we show that the Biot-Savart law can be reduced to a line integral, function only of the topology of the knot type, through the winding number, and the toroidal geometry. Since in ideal conditions the knot dynamics is completely controlled by the Biot-Savart law, we can analyze the effects of the winding number on the self-induced velocity. This study is carried out by applying the de-singularization prescription of Moore-Saffman [3] and we present some preliminary results. This study provides also important information as regards the generic role of twist in relation to the the long-term behaviour of helical structures and the kinetic helicity of complex vortex flows [4].

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A STYDY OF STATIONARY VORTEX PATCHES

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The use of the Schwarz function to investigate the dynamics [1] and the stationary shapes [2, 3] of uniform vortices is a rather old and powerful approach [4]. Along the same way, a general approach to find the boundary (∂P) of a uniform vortex which rotates without changing its shape at constant angular velocity Ω is here proposed.

In a previous paper, the tangent derivative of the conjugate of the complex velocity on the patch boundary has been investigated [5]. It has been shown that such a derivative is related to the tangent derivative of the Schwarz function $\boldsymbol{\Phi}(\boldsymbol{x}) = \overline{\boldsymbol{x}}$ of the curve ∂P. Starting from the above mentioned paper, the conjugate of the complex velocity on the patch boundary follows, through an integration by parts, as:

$$\overline{\boldsymbol{u}}(\boldsymbol{x}) = -\frac{\mathrm{i}}{2}\left[\frac{1}{2\pi\mathrm{i}}\fint_{\partial P} d\boldsymbol{y}\,\frac{\boldsymbol{\Phi}(\boldsymbol{y})}{\boldsymbol{x}-\boldsymbol{y}} + \frac{1}{2}\,\boldsymbol{\Phi}(\boldsymbol{x})\right] . \tag{1}$$

As a sample case, consider an elliptical vortex, with semi-axes a along x and $b\,(< a)$ along y. The focal distance $2c = 2\sqrt{a^2 - b^2}$ and the quantities $\alpha = (a^2 + b^2)/c^2$, $\beta = 2ab/c^2$ will be also used. The Schwarz function of the boundary of the elliptical patch is given by:

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \alpha\boldsymbol{x} - \beta\sqrt{\boldsymbol{x}^2 - c^2}\,. \tag{2}$$

By inserting the function (2) into the equation (1), the conjugate of the velocity along the boundary follows:

$$\overline{\boldsymbol{u}}(\boldsymbol{x}) = \frac{\mathrm{i}\beta}{2}\left(\sqrt{\boldsymbol{x}^2 - c^2} - \boldsymbol{x}\right) .$$

The elliptical vortex is a stationary solution of the Euler equations, as stated by Kirchhoff [6]. The angular velocity Ω follows from the eigenvalue problem:

$$\mathrm{Re}\left\{\boldsymbol{\tau}\left[\frac{1}{2\pi\mathrm{i}}\fint_{\partial P} d\boldsymbol{y}\,\frac{\boldsymbol{\Phi}(\boldsymbol{y})}{\boldsymbol{x}-\boldsymbol{y}} + \frac{\boldsymbol{\Phi}}{2}\right]\right\} \equiv 2\Omega\,\mathrm{Re}(\boldsymbol{\tau}\boldsymbol{\Phi})\,, \tag{3}$$

in which $\boldsymbol{\tau}$ is a vector tangent to the boundary ∂P and $\mathrm{Re}(\boldsymbol{x})$ is the real part of the complex number $\boldsymbol{x}$. By inserting the Schwarz function (2) into the condition (3), one obtains the well known result $\Omega = ab/(a+b)^2$.

In the present paper, the analysis is carried out for given forms of $\boldsymbol{\Phi}$, having a suitable set of free parameters. In particular, the class of the Schwarz functions:

$$\boldsymbol{\Phi}(\boldsymbol{z}) = \sum_{i=1}^{n} \frac{\boldsymbol{a}_i}{\boldsymbol{z} - \boldsymbol{z}_i}\,, \tag{4}$$

$\boldsymbol{z}$ being a point on the unit circle and n a small positive integer, will be considered. The constraints on both residuals $\boldsymbol{a}_i$ and poles $\boldsymbol{z}_i$ of the function (4) for a stationary, uniform vortex are investigated.

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ABOUT SOLVABILITY AND NUMERICAL SIMULATION OF NONSTATIONARY FLOW OF IDEAL FLUID WITH A FREE BOUNDARY

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Let an inviscid incompressible fluid occupy a domain in the plane (x,y) bounded by the free surface $-\infty < y \leqslant \eta(x,t)$, $-\infty < x < \infty$, $t > 0$. Assuming that the fluid flow is potential, we have $v(x,y,t) = \nabla\Phi(x,y,t)$, where $v(x,y,t)$ is a two-dimensional velocity field and $\Phi(x,y,t)$ is a potential.

The incompressibility condition $div\ v = 0$ implies that the velocity potential obeys the Poisson equation

$$\Delta\Phi(x,y,t) = 0. \tag{1}$$

Equation (1) is supplemented with the boundary and initial conditions

$$(\eta_t + \Phi_x\eta_x - \Phi_y)|_{y=\eta(x,t)} = 0,$$
$$(\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + gy)|_{y=\eta(x,t)} = 0,$$
$$\Phi_y|_{y=-\infty} = 0,$$

where g is the acceleration of gravity.

We consider equivalent equations, called the Dyachenko's equations, describing nonstationary motion of ideal liquid with free boundary in a gravitational field. Dyachenko's equations are nonlinear integro-differential equations. They turn out to be convenient for numerical modeling.

Existence of analytic solutions of the above equations for a sufficiently small time interval is proved. Solutions from Sobolev spaces of finite order are also investigated.

In the second part of the work, a numerical method for obtaining approximate solutions is constructed. The convergence is proved, provided that a smooth solution exists. An efficient numerical scheme is proposed.

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SYMMETRIC PAIRS OF POINT VORTICES INTERACTING WITH A NEUTRALLY BUOYANT 2D CIRCULAR CYLINDER

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Introduction

The dynamic interaction of N symmetric pairs of point vortices with a neutrally buoyant 2D rigid circular cylinder in the inviscid Hamiltonian model of Shashikanth, Marsden, Burdick and Kelly (SMBK) [1] and Shashikanth [2] is examined. A schematic sketch is shown in Figure 1. The cylinder moves freely along a straight line under the pressure field induced on its surface by the flow. The model may be thought of as a section of an inviscid axisymmetric model of a neutrally buoyant sphere interacting with N coaxial circular vortex rings and has applications to problems such as fish swimming. The Hamiltonian structure of this half-space model is first presented. The cases $N = 1$ and $N = 2$ are then examined in detail. Equilibria, bifurcations, linear stability and phase portraits are studied and for both these cases an important bifurcation parameter involving the total linear 'momentum' of the system, the strength of the vortex pairs and the radius of the cylinder emerges.

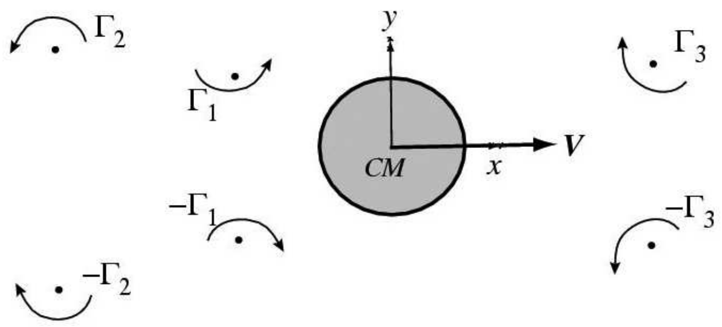

Fig 1

The symmetric half-space

The phase space of the model is

$$P_{sym} := \{p \in P \mid x_1 - x_{N+1} = 0, y_1 + y_{N+1} = 0, ... \\ ...., x_N - x_{2N} = 0, y_N + y_{2N} = 0, L_x = constant, L_y = 0\},$$

where P is a Poisson manifold and is the phase space of a rigid circular cylinder dynamically interacting with $2N$ vortices when the sum of the vortex strengths is zero SMBK model [1, 2]. P_{sym} is an invariant subspace under the flow of the SMBK Hamiltonian vector field on P. The vector $\mathbf{L} = (L_x, L_y)$ is the linear momentum variable and $(x_1, y_1, ..., x_{2N}, y_{2N})$ are the coordinates of the $2N$ vortices in the body-fixed frame. The Hamiltonian vector field of this system, tangent to level sets of $\mathbf{L}$, relative to the symplectic form $\Omega_{P_{sym}} := \sum_{j=1}^{N} 2\Gamma_j\,(dx_j \wedge dy_j)$ and the Hamiltonian function $H_{sym} : P_{sym} \longrightarrow \mathbb{R}$, which is the body+fluid system kinetic energy (minus infinite contributions), is then given by

$$\frac{dx_i}{dt} = \frac{1}{2\Gamma_i}\frac{\partial H_{sym}}{\partial y_i}, \frac{dy_i}{dt} = -\frac{1}{2\Gamma_i}\frac{\partial H_{sym}}{\partial x_i}, i = 1, ..., N$$

In the above Γ_i is the strength of the ith vortex in the half-space.

The case $N = 1$**:** Define the bifurcation parameter $\bar{L} = L_x/(\Gamma_1 R)$, where R is the radius of the cylinder. Bifurcations are seen to occur at two critical values of $\bar{L}$ which are obtained (approximately) from numerical investigations as $\bar{L}_1 = 1.744$ and $\bar{L}_2 = 2.183$ The sequence of bifurcations undergone by the system is as follows:
1.In the interval $-\infty < \bar{L} < \bar{L}_1$, there is *no* equilibirum configuration.
2. In the interval, $\bar{L}_1 \leqslant \bar{L} \leqslant \bar{L}_2$, there are *four* moving Föppl equilibrium configurations, two aft of the cylinder and two fore of the cylinder. Two of these equilibria are stable centers and the other two are unstable saddles.
3. In the interval, $\bar{L}_2 < \bar{L} < \infty$, *two more* equilibrium configurations—the moving normal line equilibria— appear. One is a stable center and the other an unstable saddle. The vortices are located on the $\bar{y}$-axis in these configurations. Thus, there are a total of *six* equilibrium configurations in this interval.

The case $N = 2$**:** Two special cases of this problem yield equilibrium configurations, one in which $\Gamma_1 = \Gamma_2 = \Gamma$ and the other in which $\Gamma_1 = -\Gamma_2 = \Gamma$, and in both of which the vortices are symmetrically located about the y-axis.

Same-signed symmetric equilibria: Bifurcations are seen to occur at three critical values of $\bar{L}$ which are obtained (approximately) from numerical investigations as $\bar{L}_1 = 2.42$, $\bar{L}_2 = 3.277$ and $\bar{L}_3 = 5.529$. The sequence of bifurcations undergone by the system is as follows:
1. In the interval, $-\infty < \bar{L} \leqslant \bar{L}_1$, there is *no* equilibirum configuration.
2. In the interval, $\bar{L}_1 < \bar{L} \leqslant \bar{L}_2$, there are *two* equilibrium configurations.
3. In the interval, $\bar{L}_2 < \bar{L} \leqslant \bar{L}_3$, *two more* equilibrium configurations appear. Thus, there are a total of *four* equilibrium configurations in this interval.
4. Finally, in the interval $\bar{L}_3 < \bar{L} < \infty$,there are again only *two* equilibrium configurations.
Linear stability analysis shows that the eigenvalues associated with these equilibria are real pairs at sampled points on the equilibrium curve and thus these equilbria are unstable.

Opposite-signed symmetric equilibria: Bifurcations are seen to occur at one critical value of the parameter $\bar{L}$ obtained (approximately) from numerical investigations as $\bar{L}_1 = 0.3535$. The sequence of bifurcations undergone by the system is as follows:
1. In the interval, $-\infty < \bar{L} \leqslant \bar{L}_1$, there is *no* equilibirum configuration.
2. In the interval, $\bar{L}_1 < \bar{L} < \infty$, there is only *one* equilibrium configuration.
Linear stability analysis at sampled points on the equilibrium curve again show eigenvalues that are real pairs and hence these equilibria are unstable.

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